LIFTING DERIVATIONS AND $n-$WEAK AMENABILITY OF THE SECOND DUAL OF A BANACH ALGEBRA

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Abstract. We show that for $n \geq 2$, $n-$weak amenability of the second dual $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ implies that of $\mathcal{A}$. We also provide a positive answer for the case $n = 1$, which sharpens some older results. Our method of proof also provides a unified approach to give short proofs for some known results in the case where $n = 1$.

The concept of n-weak amenability was initiated and intensively developed by Dales, Ghahramani and Gronbæk [3]. A Banach algebra $\mathcal{A}$ is said to be n-weakly amenable ($n \in \mathbb{N}$) if every (bounded) derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ (the n^{th} dual of $\mathcal{A}$) is inner. Trivially, 1-weak amenability is nothing else than weak amenability, which was first introduced and intensively studied by Bade, Curtis and Dales [2] for commutative Banach algebras and then by Johnson [9] for a general Banach algebra.

We equip the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ with its first Arens product and focus on the following question which is of special interest, especially for the case when $n = 1$.

Does $n-$weakly amenability of $\mathcal{A}^{**}$ force $\mathcal{A}$ to be $n-$weakly amenable?

In the present paper first we shall show:

Theorem 1. *The answer to the above question is positive for any $n \geq 2$.*

Then we consider the case $n = 1$, which is a long-standing open problem with a slightly different feature from that of $n \geq 2$. This case has been investigated and partially answered by many authors (see Theorem 6, in which we rearrange some known answers from [5, 6, 7, 8]). As a consequence of our general method of proof (for the case $n = 1$), we present the next positive answer; in which, π denotes the product of $\mathcal{A}$, $\pi^* : \mathcal{A}^* \times \mathcal{A} \to \mathcal{A}^*$ is defined by

$$\langle \pi^*(a^*, a), b\rangle = \langle a^*, \pi(a, b)\rangle, \quad (a^* \in \mathcal{A}^*, a, b \in \mathcal{A}),$$

and $Z_\ell(\pi^*)$ is the left topological centre of π^*, (see the next section).

Theorem 2. *Let $\mathcal{A}$ be a Banach algebra such that every derivation $D : \mathcal{A} \to \mathcal{A}^*$ satisfies $D^{**}(\mathcal{A}^{**}) \subseteq Z_\ell(\pi^*)$. Then weak amenability of $\mathcal{A}^{**}$ implies that of $\mathcal{A}$.*

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As a rapid consequence we get the next result, part (ii) of which sharpens [4, Corollary 7.5] and also [5, Theorem 2.1] (note that $WAP(\mathcal{A}) \subseteq \mathcal{A}^* \subseteq Z_\ell(\pi^*)$); indeed, it shows that the hypothesis of Arens regularity of $\mathcal{A}$ in [4, Corollary 7.5] is superfluous.

Corollary 3. *For a Banach algebra $\mathcal{A}$, in either of the following cases, the weak amenability of $\mathcal{A}^{**}$ implies that of $\mathcal{A}$.*
(i) *If π^* is Arens regular.*
(ii) *If every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact.*

The influence of the impressive paper [7] of Ghahramani *et al.* on our work should be evident. It should finally be remarked that part (ii) of Corollary 3 actually demonstrates what Ghahramani *et al.* claimed in a remark following [7, Theorem 2.3]. Indeed, as we shall see in the proof of Theorem 2, ${J_0}^* \circ D^{**}$ is a derivation ($J_0 : \mathcal{A} \to \mathcal{A}^{**}$ denotes the canonical embedding), however they claimed that D^{**} is a derivation and in their calculation of limits they used the Arens regularity of $\mathcal{A}$; see also a remark just after the proof of [4, Corollary 7.5].

The proofs

To prepare the proofs, let us first fix some notations and preliminaries. Following the seminal work [1] of Arens, every bounded bilinear map $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ (on normed spaces) has two natural but, in general, different extensions f^{***} and f^{r***r} from $\mathcal{X}^{**} \times \mathcal{Y}^{**}$ to $\mathcal{Z}^{**}$. Here the flip map f^r of f is defined by $f^r(y,x) = f(x,y)$, the adjoint $f^* : \mathcal{Z}^* \times \mathcal{X} \to \mathcal{Y}^*$ of f is defined by

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x,y)\rangle \qquad (x \in \mathcal{X}, y \in \mathcal{Y} \text{ and } z^* \in \mathcal{Z}^*);$$

and also the second and third adjoints f^{**} and f^{***} of f are defined by $f^{**} = (f^*)^*$ and $f^{***} = (f^{**})^*$, respectively. Continuing this process one can define the higher adjoints $f^{(n)}$, $(n \in \mathbb{N})$.

We also define the left topological centre $Z_\ell(f)$ of f by

$$Z_\ell(f) \quad = \quad \{x^{**} \in \mathcal{X}^{**}; y^{**} \longrightarrow f^{***}(x^{**}, y^{**}) : \mathcal{Y}^{**} \longrightarrow \mathcal{Z}^{**} \text{ is } w^* - w^* - \text{continuous}\}.$$

A bounded bilinear mapping f is said to be Arens regular if $f^{***} = f^{r***r}$, or equivalently $Z_\ell(f) = \mathcal{X}^{**}$.

It should be remarked that, in the case where π is the multiplication of a Banach algebra $\mathcal{A}$, then π^{***} and π^{r***r} are actually the first and second Arens products on $\mathcal{A}^{**}$, respectively. From now on, we only deal with the first Arens product $\Box$ and our results are based on $(A^{**}, \Box)$. Similar results can be derived if one uses the second Arens product instead of the first one.

Consider $\mathcal{A}$ as a Banach $\mathcal{A}$-module equipped with its own multiplication π. Then $(\pi^{r*r}, \mathcal{A}^*, \pi^*)$ is the natural dual Banach $\mathcal{A}-$module, in which, π^{r*r} and π^* denote its left and right module actions, respectively. Similarly, the n^{th} dual $\mathcal{A}^{(n)}$ of $\mathcal{A}$ can be made into a Banach $\mathcal{A}-$module in a natural fashion. A direct verification reveals that $(\pi^{(3n)}, \mathcal{A}^{(2n)}, \pi^{(3n)})$ is a Banach $\mathcal{A}^{**}-$module. It induces the natural dual Banach $\mathcal{A}^{**}-$module $(\pi^{(3n)r*r}, \mathcal{A}^{(2n+1)}, \pi^{(3n+1)})$ which will be used in the sequel. Note that we have also $(\pi^{r*r(3n)}, \mathcal{A}^{(2n+1)}, \pi^{(3n+1)})$ as a Banach $\mathcal{A}^{**}-$module that induced by $(\pi^{r*r}, \mathcal{A}^*, \pi^*)$. It should be mentioned that these two actions on $\mathcal{A}^{(2n+1)}$ do not coincide, in general. For more information on the equality of these actions in the case where $n = 1$ see [4, 10].

From now on, we identify (an element of) a normed space with its canonical image in its second dual; however, we also use $J_n : \mathcal{A}^{(n)} \to \mathcal{A}^{(n+2)}$ for the canonical embedding.

We commence with the next lemma.

Lemma 4. *Let $\mathcal{A}$ be a Banach algebra, $n \in \mathbb{N}$ and let $D : \mathcal{A} \to \mathcal{A}^{(2n-1)}$ be a derivation.*

(i) If $n \geq 2$ then $[(J_{2n-2})^* \circ D^{**}] : \mathcal{A}^{**} \to \mathcal{A}^{(2n+1)}$ is a derivation.

(ii) For $n = 1$, $[J_0{}^* \circ D^{**}] : \mathcal{A}^{**} \to \mathcal{A}^{***}$ is a derivation if and only if $\pi^{***r*}(D^{**}(\mathcal{A}^{**}), \mathcal{A}) \subseteq \mathcal{A}^*$.

Proof. (i). It is enough to show that for any $a^{**}, b^{**} \in A^{**}$

$$[(J_{2n-2})^* \circ D^{**}](a^{**} \Box\, b^{**}) = \pi^{(3n+1)}([(J_{2n-2})^* \circ D^{**}](a^{**}), b^{**}) + \pi^{(3n)r*r}(a^{**}, [(J_{2n-2})^* \circ D^{**}](b^{**})).$$

To this end let $\{a_\alpha\}$ and $\{b_\beta\}$ be bounded nets in $\mathcal{A}$, w^*-converging to a^{**} and b^{**}, respectively. Then

$$\begin{aligned} D^{**}(a^{**} \Box b^{**}) &= w^* - \lim_\alpha w^* - \lim_\beta D(a_\alpha b_\beta) \\ &= w^* - \lim_\alpha w^* - \lim_\beta [\pi^{(3n-2)}(D(a_\alpha), b_\beta) + \pi^{(3n-3)r*r}(a_\alpha, D(b_\beta))] \\ &= \pi^{(3n+1)}(D^{**}(a^{**}), b^{**}) + \pi^{(3n-3)r*r***}(a^{**}, D^{**}(b^{**})). \end{aligned}$$

For each $a^{(2n-2)} \in \mathcal{A}^{(2n-2)}$,

$$\begin{aligned} &\langle (J_{2n-2})^*(\pi^{(3n-3)r*r***}(a^{**}, D^{**}(b^{**}))), a^{(2n-2)} \rangle = \\ &= \lim_\alpha \lim_\beta \langle D(b_\beta), \pi^{(3n-3)}(a^{(2n-2)}, a_\alpha) \rangle \\ &= \lim_\alpha \langle D^{**}(b^{**}), \pi^{(3n-3)}(a^{(2n-2)}, a_\alpha) \rangle \\ &= \lim_\alpha \langle [(J_{2n-2})^* \circ D^{**}](b^{**}), \pi^{(3n-3)}(a^{(2n-2)}, a_\alpha) \rangle \\ &= \langle [(J_{2n-2})^* \circ D^{**}](b^{**}), \pi^{(3n)}(a^{(2n-2)}, a^{**}) \rangle \\ &= \langle \pi^{(3n)r*r}(a^{**}, [(J_{2n-2})^* \circ D^{**}](b^{**})), a^{(2n-2)} \rangle. \end{aligned}$$

Since for $n \geq 2$,

$$\pi^{(3n)}(\mathcal{A}^{**}, \mathcal{A}^{(2n-2)}) = \pi^{(3n-3)}(\mathcal{A}^{**}, \mathcal{A}^{(2n-2)}) \subseteq \pi^{(3n-3)}(\mathcal{A}^{(2n-2)}, \mathcal{A}^{(2n-2)}) \subseteq \mathcal{A}^{(2n-2)}$$

(note that the same inclusion may not valid for the case $n = 1$; indeed, it holds if and only if $\pi^{***}(\mathcal{A}^{**}, \mathcal{A}) \subseteq \mathcal{A}$, or equivalently, $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$!), we get $\pi^{(3n)}(b^{**}, a^{(2n-2)}) \in \mathcal{A}^{(2n-2)}$ and so

$$\begin{aligned}
&\langle (J_{2n-2})^*(\pi^{(3n+1)}(D^{**}(a^{**}), b^{**})), a^{(2n-2)}\rangle = \\
&\qquad = \langle D^{**}(a^{**}), \pi^{(3n)}(b^{**}, a^{(2n-2)})\rangle \\
&\qquad = \langle [(J_{2n-2})^* \circ D^{**}](a^{**}), \pi^{(3n)}(b^{**}, a^{(2n-2)})\rangle \\
&\qquad = \langle \pi^{(3n+1)}([(J_{2n-2})^* \circ D^{**}](a^{**}), b^{**}), a^{(2n-2)}\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
&[(J_{2n-2})^* \circ D^{**}](a^{**}\Box b^{**}) = \\
&\qquad = (J_{2n-2})^*(\pi^{(3n+1)}(D^{**}(a^{**}), b^{**})) + (J_{2n-2})^*(\pi^{(3n-3)r*r***}(a^{**}, D^{**}(b^{**}))) \\
&\qquad = \pi^{(3n+1)}([(J_{2n-2})^* \circ D^{**}](a^{**}), b^{**}) + \pi^{(3n)r*r}(a^{**}, [(J_{2n-2})^* \circ D^{**}](b^{**}));
\end{aligned}$$

as required.

For (ii), examining the above proof for the case $n = 1$ shows that, ${J_0}^* \circ D^{**} : \mathcal{A}^{**} \to \mathcal{A}^{***}$ is a derivation if and only if

$${J_0}^*(\pi^{****}(D^{**}(a^{**}), b^{**})) = \pi^{****}([{J_0}^* \circ D^{**}](a^{**}), b^{**}) \quad (a^{**}, b^{**} \in \mathcal{A}^{**}),$$

which holds if and only if

$$\langle \pi^{****}(D^{**}(a^{**}), b^{**}), a\rangle = \langle \pi^{****}([{J_0}^* \circ D^{**}](a^{**}), b^{**}), a\rangle \qquad (a \in \mathcal{A});$$

or equivalently,

$$\langle \pi^{***r*}(D^{**}(a^{**}), a), b^{**}\rangle = \langle \pi^{***r*}([{J_0}^* \circ D^{**}](a^{**}), a), b^{**}\rangle.$$

As $\pi^{***r*}([{J_0}^* \circ D^{**}](a^{**}), a) = \pi^{**r}([{J_0}^* \circ D^{**}](a^{**}), a) \in \mathcal{A}^*$ and also $\pi^{***r*}(D^{**}(a^{**}), a)_{|\mathcal{A}} = \pi^{**r}([{J_0}^* \circ D^{**}](a^{**}), a)$; the map $[{J_0}^* \circ D^{**}] : \mathcal{A}^{**} \to \mathcal{A}^{***}$ is a derivation if and only if $\pi^{***r*}(D^{**}(a^{**}), a) \in \mathcal{A}^*$, as claimed.

□

We are now ready to present the proofs of the main results.

Proof of Theorem 1. Let $n \in \mathbb{N}$, $D : \mathcal{A} \to \mathcal{A}^{(2n)}$ be a derivation and let $a^{**}, b^{**} \in A^{**}$. As $(\pi^{(3n+3)}, \mathcal{A}^{(2n+2)}, \pi^{(3n+3)})$ is a Banach $\mathcal{A}^{**}-$module, a standard double limit process

argument−similar to what has been used at the beginning of the proof of the preceding lemma− shows that $D^{**}:\mathcal{A}^{**}\to\mathcal{A}^{(2n+2)}$ satisfies

$$D^{**}(a^{**}\Box b^{**})=\pi^{(3n+3)}(D^{**}(a^{**}),b^{**})+\pi^{(3n+3)}(a^{**},D^{**}(b^{**})).$$

Therefore D^{**} is a derivation and so $(2n)-$weak amenability of $\mathcal{A}^{**}$ implies that $D^{**}=\delta_{a^{(2n+2)}}$ for some $a^{(2n+2)}\in\mathcal{A}^{(2n+2)}$. Now we get $D=\delta_{(J_{2n-1})^*(a^{(2n+2)})}$. Thus D is inner and so $\mathcal{A}$ is $(2n)-$weakly amenable.

Now for the odd case, suppose that $\mathcal{A}^{**}$ is $(2n-1)-$weakly amenable and let $D:\mathcal{A}\to\mathcal{A}^{(2n-1)}$ be a derivation. Then as we have seen in Lemma 4, when $n\geq 2$ the mapping $[(J_{2n-2})^*\circ D^{**}]:\mathcal{A}^{**}\to\mathcal{A}^{(2n+1)}$ is a derivation. But then, by the assumption, $[(J_{2n-2})^*\circ D^{**}]=\delta_{a^{(2n+1)}}$ for some $a^{(2n+1)}\in\mathcal{A}^{(2n+1)}$. It follows that $D=\delta_{(J_{2n-2})^*(a^{(2n+1)})}$, so that D is inner, as claimed. □

Proof of Theorem 2. Let $a^{**},b^{**}\in\mathcal{A}^{**},a\in\mathcal{A}$ and let $\{a^{**}_\alpha\}$ be a net in $\mathcal{A}^{**}$ w^*-converging to a^{**}. As $D^{**}(b^{**})\in Z_\ell(\pi^*)$,

$$\begin{aligned}\lim_\alpha <\pi^{***r*}(D^{**}(b^{**}),a),a^{**}_\alpha> &= \lim_\alpha <D^{**}(b^{**}),\pi^{***}(a^{**}_\alpha,a)>\\ &= \lim_\alpha <\pi^{****}(D^{**}(b^{**}),a^{**}_\alpha),a>\\ &= <\pi^{****}(D^{**}(b^{**}),a^{**}),a>\\ &= <D^{**}(b^{**}),\pi^{***}(a^{**},a)>\\ &= <\pi^{***r*}(D^{**}(b^{**}),a),a^{**}>.\end{aligned}$$

And this means that $\pi^{***r*}(D^{**}(b^{**}),a)\in\mathcal{A}^*$, so that ${J_0}^*\circ D^{**}$ is derivation by Lemma 4. Now by the assumption ${J_0}^*\circ D^{**}=\delta_{a^{***}}$, for some $a^{***}\in\mathcal{A}^{***}$, and this follows that $D=\delta_{{J_0}^*(a^{***})}$, so that $\mathcal{A}$ is weakly amenable. □

Further consequences

Recall that for a derivation $D:\mathcal{A}\to\mathcal{A}^*$ the second adjoint D^{**} is a derivation if and only if $\pi^{r*r***}(a^{**},D^{**}(b^{**}))=\pi^{***r*r}(a^{**},D^{**}(b^{**}))$, for every $a^{**},b^{**}\in\mathcal{A}^{**}$; or equivalently, $\pi^{****}(D^{**}(\mathcal{A}^{**}),\mathcal{A}^{**})\subseteq\mathcal{A}^*$; see [4, Theorem 7.1] and also [10, Theorem 4.2] for a more general case. While, as Lemma 4 demonstrates, ${J_0}^*\circ D^{**}$ is a derivation if and only if $\pi^{***r*}(D^{**}(\mathcal{A}^{**}),\mathcal{A})\subseteq\mathcal{A}^*$. In the next result we investigate the interrelation between D^{**} and ${J_0}^*\circ D^{**}$.

Proposition 5. *Let $D:\mathcal{A}\to\mathcal{A}^*$ be a derivation.*

(i) If D^{**} is a derivation and $\mathcal{A}^{**}\Box\mathcal{A}=\mathcal{A}^{**}$ then ${J_0}^*\circ D^{**}$ is a derivation.

(ii) If ${J_0}^*\circ D^{**}$ is a derivation and $\mathcal{A}$ is Arens regular then D^{**} is a derivation.

Proof. (i). As $\mathcal{A}^{**}\Box\mathcal{A} = \mathcal{A}^{**}$, for each $b^{**} \in \mathcal{A}^{**}$ there exist $a^{**} \in \mathcal{A}^{**}$ and $a \in \mathcal{A}$ such that $a^{**}\Box a = b^{**}$. Then

$$\begin{aligned}\pi^{***r*}(D^{**}(b^{**}), b) &= \pi^{***r*}(D^{**}(a^{**}\Box a), b)\\ &= \pi^{***r*}(\pi^{****}(D^{**}(a^{**}), a) + \pi^{***r*r}(a^{**}, D(a)), b)\\ &= \pi^{r*}(\pi^{****}(D^{**}(a^{**}), a) + \pi^{**}(a^{**}, D(a)), b) \in \mathcal{A}^*.\end{aligned}$$

It follows from Lemma 4 that ${J_0}^* \circ D^{**}$ is a derivation.

(ii). Since ${J_0}^* \circ D^{**}$ is a derivation,

$$ {J_0}^*(\pi^{****}(D^{**}(a^{**}), b^{**})) = \pi^{****}([{J_0}^* \circ D^{**}](a^{**}), b^{**}) \quad (a^{**}, b^{**} \in \mathcal{A}^{**}).$$

Let $\{a_\alpha\}$ be a bounded net in $\mathcal{A}$, w^*-converging to a^{**}. Then as $\mathcal{A}$ is Arens regular,

$$\begin{aligned}\langle \pi^{r*r***}(a^{**}, D^{**}(b^{**})), c^{**}\rangle &= \lim_\alpha \langle \pi^{r*r**}(D^{**}(b^{**}), c^{**}), a_\alpha\rangle\\ &= \lim_\alpha \langle {J_0}^*(\pi^{****}(D^{**}(b^{**}), c^{**})), a_\alpha\rangle\\ &= \lim_\alpha \langle \pi^{****}([{J_0}^* \circ D^{**}](b^{**}), c^{**}), a_\alpha\rangle\\ &= \lim_\alpha \langle [{J_0}^* \circ D^{**}](b^{**}), \pi^{***}(c^{**}, a_\alpha)\rangle\\ &= \langle [{J_0}^* \circ D^{**}](b^{**}), \pi^{***}(c^{**}, a^{**})\rangle\\ &= \langle \pi^{***r*r}(a^{**}, D^{**}(b^{**})), c^{**}\rangle,\end{aligned}$$

for all $c^{**} \in \mathcal{A}^{**}$. Therefore D^{**} is a derivation. $\Box$

As a by-product of our method of proof we provide a unified approach to new proofs for some known results for the case where $n = 1$.

Theorem 6. *In either of the following cases, weak amenability of $\mathcal{A}^{**}$ implies that of $\mathcal{A}$.*

(i) *$\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$;* [7, Theorem 2.3].

(ii) *$\mathcal{A}$ is a dual Banach algebra;* [6, Theorem 2.2].

(iii) *$\mathcal{A}$ is a right ideal in $\mathcal{A}^{**}$ and $\mathcal{A}^{**}\Box\mathcal{A} = \mathcal{A}^{**}$;* [5, Theorem 2.4].

Proof. In either cases, it sufficients to show that for a derivation $D : \mathcal{A} \to \mathcal{A}^*$ the map ${J_0}^* \circ D^{**} : \mathcal{A}^{**} \to \mathcal{A}^{***}$ is also a derivation, or equivalently, $\pi^{***r*}(D^{**}(\mathcal{A}^{**}), \mathcal{A}) \subseteq \mathcal{A}^*$.

(i) If $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$, i.e. $\mathcal{A}^{**}\Box\mathcal{A} \subseteq \mathcal{A}$, then for each $a^{**}, b^{**} \in A^{**}, a \in \mathcal{A}$,

$$\begin{aligned}\langle \pi^{***r*}(D^{**}(a^{**}), a), b^{**}\rangle = \langle D^{**}(a^{**}), b^{**}\Box a\rangle &= \langle \pi^{***r*}([{J_0}^* \circ D^{**}](a^{**}), a), b^{**}\rangle\\ &= \langle \pi^{**r}([{J_0}^* \circ D^{**}](a^{**}), a), b^{**}\rangle.\end{aligned}$$

Therefore $\pi^{***r*}(D^{**}(a^{**}), a) = \pi^{**r}([{J_0}^* \circ D^{**}](a^{**}), a) \in \mathcal{A}^*$, as required.

(ii) Let $\mathcal{A}$ be a dual Banach algebra with a predual $\mathcal{A}_*$. It is easy to verify that ${J_0}^* \circ D^{**} = D \circ (J_{\mathcal{A}_*})^*$, where $J_{\mathcal{A}_*} : \mathcal{A}_* \to \mathcal{A}^*$ denotes the canonical embedding. Now using the fact that $(J_{\mathcal{A}_*})^* : \mathcal{A}^{**} \to \mathcal{A}$ is a homomorphism, a direct verification shows that $D \circ (J_{\mathcal{A}_*})^*$ is a derivation.

(iii) To show that ${J_0}^* \circ D^{**} : \mathcal{A}^{**} \to \mathcal{A}^{***}$ is a derivation, by Proposition 5 we only need to show that D^{**} is a derivation. However this was done in the proof of [5, Theorem 2.4], we also give the next somewhat shorter proof for this. Let $a^{**}, b^{**}, c^{**}, d^{**} \in \mathcal{A}^{**}$ and $a \in \mathcal{A}$ such that $d^{**}\Box a = b^{**}$. As $a\Box c^{**} \in \mathcal{A}$,

$$\begin{aligned}
\langle \pi^{****}(D^{**}(a^{**}), b^{**}), c^{**}\rangle &= \langle \pi^{****}(D^{**}(a^{**}), d^{**}\Box a), c^{**}\rangle \\
&= \langle \pi^{****}(D^{**}(a^{**}), d^{**}), a\Box c^{**}\rangle \\
&= \langle \pi^{*}({J_0}^*(\pi^{****}(D^{**}(a^{**}), d^{**})), a), c^{**}\rangle.
\end{aligned}$$

We have thus $\pi^{****}(D^{**}(a^{**}), b^{**}) = \pi^{*}({J_0}^*(\pi^{****}(D^{**}(a^{**}), d^{**})), a) \in \mathcal{A}^*$, and this says that $\pi^{****}(D^{**}(\mathcal{A}^{**}), \mathcal{A}^{**}) \subseteq \mathcal{A}^*$, as required. $\square$

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